

Improvements to Turing's Method II.

T. S. Trudgian*

Mathematical Sciences Institute

The Australian National University, ACT 0200, Australia

timothy.trudgian@anu.edu.au

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Abstract

Turing's method uses explicit bounds on $|\int_{t_1}^{t_2} S(t) dt|$, where $\pi S(t)$ is the argument of the Riemann zeta-function. This article improves the bound on $|\int_{t_1}^{t_2} S(t) dt|$ given in [8].

1 Introduction

Let $\zeta(s)$ be the Riemann zeta-function, and let $N(T)$ denote the number of zeroes of $\zeta(s)$ with $0 < \Re(s) < 1$ and $0 < \Im(s) < T$. One seeks to calculate $N(T)$ as follows.

First one finds zeroes by locating sign changes of a real-valued function the zeroes of which agree with the non-trivial zeroes of the zeta-function. This gives one a lower bound on the number of zeroes of $\zeta(s)$ with $0 < \Im(s) < T$.

To check whether this initial analysis has omitted some zeroes one employs Turing's method. This was first announced by Turing [11] in 1953 and has been used extensively since then. Recently, another method has been deployed by Büthe [2].

To apply Turing's method one needs good explicit bounds on

$$\left| \int_{t_1}^{t_2} S(t) dt \right|,$$

for $t_2 > t_1 > 0$, where $\pi S(t)$ is defined to be the argument of $\zeta(\frac{1}{2} + it)$. For a complete definition and a brief history of the problem, see [8, §1] and [4, Ch. 7].

This article improves [8] and contains frequent references to the results therein. The main result is

Theorem 1.

$$\left| \int_{t_1}^{t_2} S(t) dt \right| \leq 1.698 + 0.183 \log \log t_2 + 0.049 \log t_2, \quad (1)$$

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for $t_2 > t_1 > 10^5$. If the right-side of (1) is replaced by $a + b \log \log t_2 + c \log t_2$, one may use Table 1 on page 6 for more specific values of a, b and c .

In [8] the main result followed from Lemma 2.8 and Lemma 2.11 which concerned respectively obtaining an upper and a lower bound for $\Re \log \zeta(s)$ for $\Re(s) \geq \frac{1}{2}$. This article refines only the upper bound. Theorem 1 improves on Theorem 2.2 in [8] for all $t_2 \geq 10^5$.

The idea in this article is to use more sophisticated estimates on $\zeta(\sigma + it)$ for $\frac{1}{2} \leq \sigma \leq 1$; these estimates have been given in [7] and [10]. A bound on $|\zeta(s)|$ is given in §2, a proof of Theorem 1 is given in §3, and some concluding remarks are provided in §4.

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2 Bounding $|\zeta(\sigma + it)|$ across the strip $\frac{1}{2} \leq \sigma \leq 1 + \delta$

Using the inequality $\log(1 + x) \leq x$ it is easy to see that

$$\log |Q_0 + \sigma + it| - \log t \leq \frac{1}{2} \left(\frac{\sigma_1 + Q_0}{t_0} \right)^2, \quad (2)$$

for $\sigma \leq \sigma_1$ and $t \geq t_0$ and any $Q_0 \geq 0$. With the trivial observations $\log |Q_0 + \sigma + it| \geq \log t$ and $|\arg(Q_0 + \sigma + it)| \leq \frac{\pi}{2}$ at hand, we may apply (2) to see that

$$|\log(Q_0 + \sigma + it)| \leq (1 + a_0) \log t, \quad (\sigma \leq \sigma_1) \quad (3)$$

where

$$a_0 = a_0(\sigma_1) = \frac{\sigma_1 + Q_0}{2t_0^2 \log t_0} + \frac{\pi}{2 \log t_0} + \frac{\pi(\sigma_1 + Q_0)^2}{4t_0 \log^2 t_0}.$$

Suppose that

$$|\zeta(\frac{1}{2} + it)| \leq k_1 t^{k_2} (\log t)^{k_3}, \quad (t \geq t_1), \quad |\zeta(1 + it)| \leq k_4 \log t^{k_5}, \quad (t \geq t_2). \quad (4)$$

Consider the function $h(s) = (s - 1)\zeta(s)$, which is entire. Once we are able to exhibit bounds for $|h(s)|$ using the information in (4) we can apply a version of the Phragmén–Lindelöf principle to bound $|\zeta(s)|$. Using Lemma 3 in [9] and (3) and (2) we may prove

Lemma 1. *Let $h(s) = (s - 1)\zeta(s)$, and let δ be a positive real number. Furthermore, let $Q_0 \geq 0$ be a number for which*

$$\begin{aligned} |h(\frac{1}{2} + it)| &\leq k_1 |Q_0 + \frac{1}{2} + it|^{k_2+1} (\log |Q_0 + \frac{1}{2} + it|)^{k_3} \\ |h(1 + it)| &\leq k_4 |Q_0 + 1 + it| (\log |Q_0 + 1 + it|)^{k_5} \\ |h(1 + \delta + it)| &\leq \zeta(1 + \delta) |Q_0 + 1 + \delta + it|, \end{aligned}$$

for all t . Then for $\sigma \in [\frac{1}{2}, 1]$ and $t \geq t_0$,

$$|\zeta(s)| \leq \alpha_1 k_1^{2(1-\sigma)} k_4^{2(\sigma-\frac{1}{2})} t^{2k_2(1-\sigma)} (\log t)^{2(k_3(1-\sigma)+k_5(\sigma-\frac{1}{2}))}, \quad (5)$$

where

$$\alpha_1 = (1 + a_1(1 + \delta, Q_0, t_0))^{k_2+1} (1 + a_0(1 + \delta, Q_0, t_0))^{k_3+k_5},$$

and

$$a_0(\sigma, Q_0, t) = \frac{\sigma + Q_0}{2t^2 \log t} + \frac{\pi}{2 \log t} + \frac{\pi(\sigma + Q_0)^2}{4t \log^2 t}, \quad a_1(\sigma, Q_0, t) = \frac{\sigma + Q_0}{t}.$$

Whereas for $\sigma \in [1, 1 + \delta]$ and $t \geq t_0$,

$$|\zeta(s)| \leq \alpha_2 k_4^{\frac{1+\delta-\sigma}{\delta}} \zeta(1 + \delta)^{\frac{\sigma-1}{\delta}} (\log t)^{k_5(\frac{1+\delta-\sigma}{\delta})}, \quad (6)$$

where

$$\alpha_2 = (1 + a_1(1 + \delta, Q_0, t_0))(1 + a_0(1 + \delta, Q_0, t_0))^{k_5}.$$

Finally, for all $\sigma \in [\frac{1}{2}, 1 + \delta]$ and $t \geq t_0$ we have

$$|\zeta(s)| \leq (1 + a_1(1 + \delta, Q_0))^{k_2+1} (1 + a_0(1 + \delta, Q_0))^{k_3+k_5} k_1 t^{k_2} (\log t)^{k_3},$$

provided that

$$t^{k_2} (\log t)^{k_3-k_5} \geq \frac{k_4}{k_1}, \quad t \geq \exp \left\{ \left(\frac{\zeta(1 + \delta)}{k_4} \right)^{\frac{1}{k_5}} \right\}. \quad (7)$$

Proof. In applying Lemma 3 of [9] to $h(s)$ we need to relate $|Q_0 + s|$ and $|s - 1|$ to t . We simply note that

$$\left| \frac{Q_0 + s}{s - 1} \right| \leq \frac{|Q_0 + s|}{t} \leq 1 + \frac{\sigma + Q_0}{t_0} = 1 + a_1(\sigma, Q_0) \leq 1 + a_1(1 + \delta, Q_0),$$

in both regions $\sigma \in [\frac{1}{2}, 1]$ and $\sigma \in [1, 1 + \delta]$. Since a_0 and a_1 are small for any respectable value of t_0 we throw away some information in the exponents of $1 + a_1$ and $1 + a_0$. For example, in proving (5) we arrive at

$$(1 + a_1)^{2k_2(1-\sigma)+1} (1 + a_0)^{2k_3(1-\sigma)+2k_5(\sigma-\frac{1}{2})}.$$

Rather than retain this dependence on σ in the exponents, we simply bound $1 - \sigma$ and $\sigma - \frac{1}{2}$ by $\frac{1}{2}$. A similar procedure is applied to prove (6).

To prove the bound in the region $\sigma \in [\frac{1}{2}, 1 + \delta]$ we note that the bounds in (5) and (6) are decreasing in σ if the inequalities in (7) are met. Finally, the bound in (5), evaluated at $\sigma = \frac{1}{2}$ exceeds the bound in (6), evaluated at $\sigma = 1$. This completes the lemma. \square

It is worth recording values of k_1, \dots, k_5 , which we do in

Corollary 1. For $\sigma \in [\frac{1}{2}, 1 + \delta]$ and $t \geq t_0$ we have

$$|\zeta(s)| \leq 0.732(1 + a_1(1 + \delta, 5, t_0))^{7/6}(1 + a_0(1 + \delta, 5, t_0))^2 t^{1/6} \log t,$$

provided that

$$t \geq \max\{1.16, \exp[4\zeta(1 + \delta)/3]\}.$$

Proof. In [10] it was shown that $|\zeta(1 + it)| \leq \frac{3}{4} \log t$ for $t \geq 3$. In [7] it was shown that $|\zeta(\frac{1}{2} + it)| \leq 0.732|4.678 + it|^{\frac{1}{6}} \log |4.678 + it|$ for all t . One may therefore choose

$$(k_1, k_2, k_3, k_4, k_5, Q_0) = (0.732, \frac{1}{6}, 1, \frac{3}{4}, 1, 5) \quad (8)$$

in Lemma 1, which proves the corollary. \square

Although we shall use (8) in our computation we proceed with the variables (k_1, \dots, Q_0) as parameters. We remark that, although Corollary 1 is not used in this article, it is derived at very little additional cost and should prove useful for related problems.

3 Proof of Theorem 1

We are now able to proceed to the proof of Theorem 1. We need to give an upper bound for

$$I = \Re \int_{\frac{1}{2} + it}^{\infty + it} \log \zeta(s) ds.$$

To that end, we shall write

$$I \leq \int_{\frac{1}{2} + it}^{1 + it} \log |\zeta(s)| ds + \int_{1 + it}^{1 + \delta + it} \log |\zeta(s)| ds + \int_{1 + \delta}^{\infty} \log |\zeta(\sigma)| d\sigma, \quad (9)$$

and apply (5) to the first integral in (9) and (6) to the second. This gives us

Lemma 2. For $t \geq t_0$ we have

$$I \leq A_1 + B_1 \log \log t + C_1 \log t,$$

where

$$\begin{aligned} A_1 = & \int_{1 + \delta}^{\infty} \log |\zeta(\sigma)| d\sigma + \left(\frac{k_2}{4} + \frac{1}{2} + \delta\right) \log(1 + a_1(1 + \delta, Q_0, t_0)) + \frac{1}{4} \log k_1 \\ & + \left(\frac{k_3}{4} + \frac{k_5}{4} + \frac{k_5 \delta}{2}\right) \log(1 + a_0(1 + \delta, Q_0, t_0)) + \left(\frac{1}{4} + \frac{\delta}{2}\right) \log k_4 + \frac{\delta}{2} \log \zeta(1 + \delta) \end{aligned}$$

and

$$B_1 = \left(\frac{k_3 + k_5}{4} + \frac{\delta k_5}{2}\right), \quad C_1 = \frac{k_2}{4}.$$

The term corresponding to C_1 in Lemma 2.8 of [8] is, $k_2/4 + \delta k_2/2$. Since we are not permitted to take δ too small, lest the integral in A_1 become too large, this represents a considerable qualitative saving. This is due entirely to estimating $|\zeta(s)|$, not in one go over $\sigma \in [\frac{1}{2}, 1 + \delta]$ as in [8] but by using Lemma 1. We combine Lemma 2 with Lemma 2.11 in [8] to obtain

Theorem 2. *For $t_2 \geq t_1 \geq 10^5$,*

$$\left| \int_{t_1}^{t_2} S(t) dt \right| \leq a + b \log \log t_2 + c \log t_2,$$

where

$$\begin{aligned} \pi a = & \int_{1+\delta}^{\infty} \log |\zeta(\sigma)| d\sigma + \left(\frac{k_2}{4} + \frac{1}{2} + \delta\right) \log(1 + a_1(1 + \delta, Q_0, 10^5)) + \frac{1}{4} \log k_1 \\ & + \left(\frac{k_3}{4} + \frac{k_5}{4} + \frac{k_5 \delta}{2}\right) \log(1 + a_0(1 + \delta, Q_0, 10^5)) + \left(\frac{1}{4} + \frac{\delta}{2}\right) \log k_4 + \frac{\delta}{2} \log \zeta(1 + \delta) \\ & + d^2 \log 4 \left\{ -\frac{\zeta'(\frac{1}{2} + d)}{\zeta(\frac{1}{2} + d)} - \frac{1}{2} \log 2\pi + \frac{1}{4} \right\} + \frac{d^2}{2} \log \pi - \frac{1}{2} \int_{1+2d}^{\infty} \log \zeta(\sigma) d\sigma \\ & + \int_{\frac{1}{2}+d}^{\infty} \log \zeta(\sigma) d\sigma - \frac{1}{2} \int_{1+2d}^{1+4d} \log \zeta(\sigma) d\sigma + \int_{\frac{1}{2}+d}^{\frac{1}{2}+2d} \log \zeta(\sigma) d\sigma + 3 \times 10^{-4}, \end{aligned} \quad (10)$$

and

$$\pi b = \left(\frac{k_3 + k_5}{4} + \frac{\delta k_5}{2} \right), \quad \pi c = \frac{k_2}{4} + \frac{d^2}{2} (\log 4 - 1). \quad (11)$$

3.1 Computation

Before we commence an analysis of the coefficients appearing in Theorem 2 we make the following observation. One may replace the values of (k_1, k_2, k_3) in (8) by

$$(k_1, k_2, k_3) = \left(\frac{4}{(2\pi)^{\frac{1}{4}}}, \frac{1}{4}, 0 \right), \quad (12)$$

which appear in [5, Lem. 2]. The values in (12) are obtained using the approximate functional equation of $\zeta(s)$: the values in (8) are obtained using exponential sums. The value $k_2 = \frac{1}{4}$ follows from convexity theorems. We call (12) the convexity result, and (8) the sub-convexity result.

As in Theorem 2.12 in [8] no term in either (10) or (11) depends on both δ and d . We can run two one-dimensional optimisations on each of a , b and c . In Table 1 we compare the results obtained from the convexity result (C), the sub-convexity result (SC), and the coefficients in Theorem 2.2 of [8], when $t_1 \approx T$. The values of δ, d, a, b and c correspond to the sub-convexity result. We find that the sub-convexity result overtakes the convexity result when $T \geq 2.85 \times 10^{10}$, which is, just barely, beneath the height to which the Riemann hypothesis has been verified — see [6].

The values used in Theorem 1 are taken from the row $T = 10^{10}$. It should be stressed that all of the results in this table are valid for $t_1 \geq 10^5$. The stated values of a, b and c are those that are close to the best values obtainable by this method when $t_1 \approx T$.

Table 1: Comparison of bounds for $|\int_{t_1}^{t_2} S(t) dt| \leq a + b \log \log t_2 + c \log t_2$

T	Thm 2.2	C	SC	d	δ	a	b	c
10^5	2.747	2.629	2.658	0.883	0.279	1.457	0.204	0.062
10^6	2.883	2.800	2.827	0.845	0.237	1.520	0.197	0.058
10^7	3.018	2.959	2.982	0.817	0.206	1.573	0.192	0.055
10^8	3.154	3.110	3.128	0.795	0.182	1.620	0.189	0.053
10^9	3.290	3.255	3.266	0.777	0.163	1.661	0.186	0.051
10^{10}	3.426	3.395	3.398	0.762	0.148	1.698	0.183	0.049
10^{11}	3.562	3.530	3.526	0.749	0.135	1.733	0.181	0.048
10^{12}	3.698	3.663	3.649	0.738	0.124	1.764	0.179	0.047
10^{13}	3.834	3.792	3.770	0.729	0.115	1.792	0.178	0.046
10^{14}	3.969	3.919	3.887	0.720	0.107	1.820	0.177	0.046
10^{15}	4.105	4.044	4.002	0.713	0.100	1.844	0.176	0.045

4 Conclusion

It seems difficult to improve substantially on Theorem 1. Given that the improvements obtained in this paper are only modest, and since further improvements would require a lot of effort in estimating $\zeta(\frac{1}{2} + it)$ or $\zeta(1 + it)$, it seems hopeless to try to improve this part of the argument.

One could try one's luck at reducing the term $\log 4$ that appears in both (10) and (11). This comes from Lemma 4.4 in [1]. Reducing this would have a more profound influence on bounding $|\int_{t_1}^{t_2} S(t) dt|$ than better bounds for $|\zeta(s)|$.

Finally, it is worth considering Theorems 3.3 and 4.3 in [8], which relate to Dirichlet L -functions and Dedekind zeta-functions. Both of these could be improved, in line with this article, were one in possession of explicit estimates on the lines $\sigma = \frac{1}{2}$ and $\sigma = 1$. One such estimate, bounding $|L(1 + it, \chi)|$ for $L(s)$ a Dirichlet L -function, appears in [3]. It is possible that this could be used to obtain an improvement to Theorem 3.3 in [8].

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